Linear ODE's: where are we?

$$\ddot{x} + a(t)\dot{x} + b(t) = f(t)$$

→ has solution

$$C_1 u_1(t) + C_2 u_2(t) + u^*(t)$$

non-proportional solutions to corresponding homogeneous eq.

particular solution

→ If a, b constant:

→ $r^2 + ar + b = 0$

→ u_1, u_2 in terms of (real and imaginary parts of) the root(s)

- if you don't like complex numbers: <u>formulae</u>

→ u^* for a few special forms of f

→ 2^{nd} order: need two data to determine constants

- could be $\begin{cases} x(t_0) = x_0 \\ x(t_1) = x_1 \end{cases}$

- but also $\begin{cases} x(t_0) = x_0 \\ \dot{x}(t_0) = y \end{cases}$

→ Stability:
Global asymptotic stability
$\leftrightarrow$ effect of initial condition $\begin{cases} x(t_0) = x_0 \\ \dot{x}(t_0) = y \end{cases}$

vanishes as $t \to +\infty$.

This is $\Longleftrightarrow$ $\left\{\begin{array}{l} \text{the roots } r_i \\ \text{have negative} \\ \underline{\text{real parts}} \end{array}\right.$

$$r_i = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

Real part $= \begin{cases} r_i & \text{if } \sqrt{\ } \text{ real} \\ -\frac{a}{2} & \text{if } \sqrt{\text{negative}} \end{cases}$

ODE systems (1st order)

$$\dot{\bar{x}} = \bar{F}(t, \bar{x})$$

$$\bar{F} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \qquad \bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

→ The "constant of integration" will now be $\bar{C} \in \mathbb{R}^n$

Consider now the linear system

$$\dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{b}(t)$$

where we assume $\bar{A}$ a matrix of <u>constants</u> (no t-dependence).

→ Can find an n^{th} order ODE in $\mathbb{R}^2$ for each coordinate.

→ $n=2$:
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \bar{A}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$$

$$\Rightarrow \quad \ddot{x} - (\mathrm{tr}\,\bar{A})\,\dot{x} + (\det \bar{A})\,x =$$
$$= a_{12} b_2 - a_{22} b_1 + \dot{b}_1$$

char. roots: the <u>eigenvalues</u> of $\bar{A}$

ALTERNATIVE APPROACH

If
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \bar{A}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$$

($\bar{A}$ constant wrt. t)

then
$$\ddot{x} - (\mathrm{tr}\,\bar{A})\,\dot{x} + (\det \bar{A})\,x = f_1(t) \qquad (*)$$
$$\ddot{y} - (\mathrm{tr}\,\bar{A})\,\dot{y} + (\det \bar{A})\,y = f_2(t)$$

with
$$f_1(t) = a_{12}\,b_2(t) - a_{22}\,b_1(t) + \dot{b}_1(t)$$

→ If $a_{12} = 0$ then $\dot{x} = a_{11}x + b_1$

solve this, plug into
$$\underbrace{\dot{y} - a_{22}y}_{1^{st}\text{ order}} = a_{21}x + b_2$$
and solve.

→ If $a_{21} = 0$: solve first for y, then for x

Suppose $a_{12}a_{21} \neq 0$.

① Solve $(*)$ for $x = C_1u_1 + C_2u_2 + u^*$

② Then
$$y = \frac{\dot{x} - a_{11}x - b_1}{a_{12}}$$

is likely easiest! (Even if you must calculate $\dot{x}$)

Ex:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & 3\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}1\\ 2\end{pmatrix}e^{-\pi t} \qquad (\pi \approx 3.14159).$$

Eigenvalues: $\underset{\alpha}{1} \pm i \underbrace{\sqrt{3}}_{\beta}$, complex.

$$x(t) = C_1 e^{t}\cos(t\sqrt{3}) + C_2 e^{t}\sin(t\sqrt{3}) + u^*$$

where u^* solves

$$\ddot{u}^* - 2\dot{u}^* + 4u^* = 3\cdot 2e^{-\pi t} - e^{-\pi t} - \pi e^{-\pi t}$$

Fit $Ke^{-\pi t}$ to

$$(5-\pi)e^{-\pi t} = Ke^{-\pi t}(\pi^2 - 2(-\pi) + 4)$$

$$K = \frac{5-\pi}{\pi^2 + 2\pi + 4}$$

Since $\dot{x} = e^{t}\cdot(C_1\cos + C_2\sin)$

$$+ e^{t}(C_1\sqrt{3}(-\sin) + C_2\sqrt{3}(\cos))$$

$$- K\pi e^{-\pi t}$$

$$\dot{x} - \underset{=1}{a_{11}}x = [\text{these terms}] - Ke^{-\pi t}$$

$$y = \frac{\sqrt{3}}{3}e^{t}\left[C_2\cos(t\sqrt{3}) - C_1\sin(t\sqrt{3})\right] - \frac{K(1+\pi)+1}{3}e^{-\pi t}$$

!

The "$-$" for the y was wrong in the lecture.

Why is it there?

$$\lambda_1 = \alpha + i\beta$$
$$\lambda_2 = \alpha - i\beta$$

Should have the imaginary part of the __other__ eigenvalue!

(It does not matter whether you choose $\beta = \sqrt{3}$ or $\beta = -\sqrt{3}$, that will only replace "$\sin(t\sqrt{3})$" by "$-\sin t\sqrt{3}$" __both__ places — and $-C_1$ is still an arbitrary (real) constant.)

Note: if you use the form

$$Ce^{\alpha t}\cos(\omega + \beta t) \quad \text{in } x$$

you will either need both cos and sin for y, or $De^{\alpha t}\cos(\phi + \beta t)$

Special cases:

a) If $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ does not depend on t,

and $|\bar{A}| \neq 0$

then $\begin{pmatrix} u^* \\ v^* \end{pmatrix} = -\bar{A}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

* Let $\bar{x} \in \mathbb{R}^n$,

$\bar{A}$ have n distinct (complex!) eigenvalues $\lambda_1 \dots \lambda_n$ with eigenvectors $\bar{v}_1 \dots \bar{v}_n$,

- homogeneous eq has general solution
$$C_1 \bar{v}_1 e^{\lambda_1 t} + \dots + C_n \bar{v}_n e^{\lambda_n t}$$
- if $\bar{b}$ does not depend on t and $|\bar{A}| \neq 0$: particular solution $\bar{A}^{-1} \bar{b}$

Ex: $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

symmetric, has two real eigenvalues

0 and 2

eigenvector $\begin{pmatrix}1\\ -1\end{pmatrix}$ eigenvector $\begin{pmatrix}1\\ 1\end{pmatrix}$

Solution: $\underbrace{C_1 \begin{pmatrix}1\\ -1\end{pmatrix}}_{x+y\ =0\ =\dot{x},\ \text{ok};\ =\dot{y}\ \text{ok}} + C_2 e^{2t}\begin{pmatrix}1\\ 1\end{pmatrix}$

$x+y = 2C_2 e^{2t}$

$\dot{x} = \dot{y} = 2C_2 e^{2t}$

OK

Nonlinear <u>autonomous</u> systems in $\mathbb{R}^2$

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

- Constant solution at any point (x_0, y_0) such that $f(x_0, y_0) = g(x_0, y_0) = 0$.

Such is called an <u>equilibrium point</u> a.k.a. a <u>stationary state</u>.

Q's:

→ how does (x,y) behave outside eq. pts?
→ will paths converge to (x_0, y_0)?

Definition:
An eq. pt glob. asympt. stable
loc. any ....

stable: ...

we'll get to criteria for stability.

Note: "get to criteria" is a note-to-self.
Left it here rather than writing a new sheet.

Analytic tool:

The <u>Jacobian</u> of a vector transformation $\begin{pmatrix} f_1(\bar{x}) \\ \vdots \\ f_n(\bar{x}) \end{pmatrix}$

is $$\bar{J} = \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{pmatrix}$$ matrix.

To decide stability of an eq. pt $\bar{x}^*$ of $\dot{\bar{x}} = \bar{F}(\bar{x})$:

① Calculate $\bar{A} = J(\bar{x}^*)$. Matrix of <u>numbers</u>.

We have an approximation of $\bar{z}(t) = \bar{x}(t) - \bar{x}^*$:

$$\dot{\bar{z}} = \bar{A}\,\bar{z},$$ so:

② Find the eigenvalues of $\bar{A}$

. . . .

For tomorrow:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \bar{M} \begin{pmatrix} \frac{1}{2}(x^2-1) \\ y \end{pmatrix}$$

assume $|\bar{M}| \neq 0$

Nonlinear!

Jacobian: $\bar{J} = \bar{M} \begin{pmatrix} x & 1 \\ x & 1 \end{pmatrix}$

(check for yourselves!)

Eq. pts: $\bar{M} \begin{pmatrix} \frac{1}{2}(x^2-1) \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\underset{\text{since } |\bar{M}| \neq 0}{\Leftrightarrow}$ $\quad x^2 = 1$, $y = 0$ $\quad (-1,0)$ and $(1,0)$

$$\bar{J}(\pm 1, 0) = \bar{M} \begin{pmatrix} \pm 1 & 1 \\ \pm 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \pm(m_{11}+m_{12}) & m_{11}+m_{12} \\ \pm(m_{21}+m_{22}) & m_{21}+m_{22} \end{pmatrix}$$

has determinant $= 0$ (proportional columns!)

$$\begin{vmatrix} \pm p - \lambda & p \\ \pm q & q - \lambda \end{vmatrix} = \lambda^2 - \lambda(q \pm p)$$

("+" for $(1,0)$, "−" for $(-1,0)$)

eigenvalues $\lambda_1 = 0$ and $\lambda_2 = q \pm p$

$(1,0)$: unstable if $q+p > 0$, no conclusion otherwise

$(-1,0)$ unstable if $q-p > 0$, ———— " ————

Graphical tool: phase plane analysis.

$f(x,y) = 0$: curve in (x,y)-plane

$\to$ at the curve: $\dot{x} = 0$ i.e.

movement $\downarrow$ or $\uparrow$ or $\cdot$

$g(x,y) = 0$: analogous. No movement in y-direction

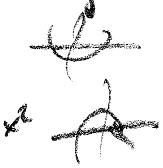
These curves are called <u>nullclines</u>.

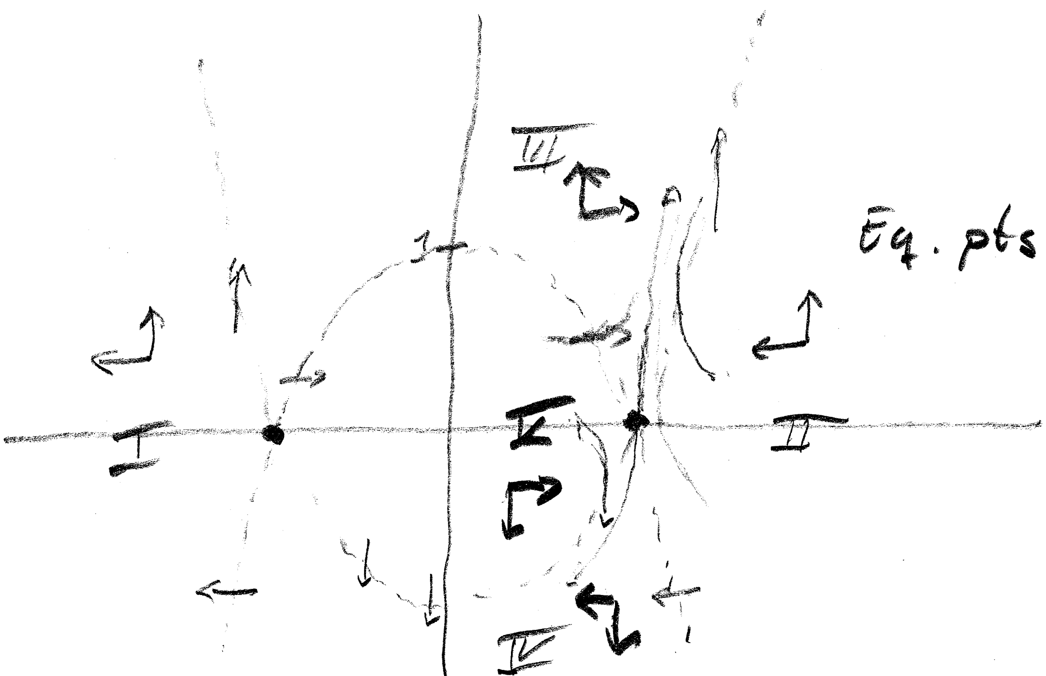
Ex:

$$\dot{x} = y - x^2 + 1$$
$$\dot{y} = y + x^2 - 1$$

zero when $y = x^2 - 1$

$y = 1 - x^2$





Eq. pts: $(\pm 1, 0)$

Nullclines partition the plane. In this case into five domains

Above	$\frown$:	$\dot{y} > 0$.	"Upwards"
Below	$- \cdot\cdot -$:		downwards
Above	$\smile$:	$\dot{x} > 0$:	right
Below	$\frown$:	$x < 0$:	left